Admissions Testing Service

STEP Mark Scheme 2015

Mathematics

STEP 9465/9470/9475

October 2015

Test

Question 1

(i)	$\frac{d}{dx}(x - \ln(1+x)) = 1 - \frac{1}{1+x}$	**B1**
	For $x > 0$, $\frac{1}{1+x} < 1$	**M1**
	Therefore $\frac{d}{dx}(x - \ln(1+x)) > 0$ for $x > 0$	**A1**
	If $x = 0$, $x - \ln(1+x) = 0$	
	Therefore $x - \ln(1+x)$ is positive for all positive x.	**B1**
	Therefore $\frac{1}{k} - \ln\left(1 + \frac{1}{k}\right) > 0$ for all positive k.	
	So, $\sum_{k=1}^{n} \frac{1}{k} > \sum_{k=1}^{n} \ln\left(1 + \frac{1}{k}\right)$	**B1**
	$\ln\left(1 + \frac{1}{k}\right) = \ln\left(\frac{k+1}{k}\right) = \ln(k+1) - \ln k$	**M1**
	So, $\sum_{k=1}^{n} \ln\left(1 + \frac{1}{k}\right) = \sum_{k=1}^{n} \ln(k+1) - \ln k = \ln(n+1) - \ln 1$	**M1**
	Therefore, $\sum_{k=1}^{n} \frac{1}{k} > \ln(n+1)$	**A1**
(ii)	$\frac{d}{dx}(x + \ln(1-x)) = 1 - \frac{1}{1-x}$	**B1**
	For $0 < x < 1$, $\frac{1}{1-x} > 1$	**M1**
	Therefore $\frac{d}{dx}(x + \ln(1-x)) < 0$ for $0 < x < 1$.	**A1**
	If $x = 0$, $x + \ln(1-x) = 0$	
	Therefore $x + \ln(1-x)$ is negative for $0 < x < 1$.	**B1**
	Therefore $\frac{1}{k^2} + \ln\left(1 - \frac{1}{k^2}\right) < 0$ for all $k > 1$.	
	So, $\sum_{k=2}^{n} \frac{1}{k^2} < \sum_{k=2}^{n} -\ln\left(1 - \frac{1}{k^2}\right)$	**B1**
	$-\ln\left(1 - \frac{1}{k^2}\right) = -\ln\left(\frac{k^2-1}{k^2}\right) = -\ln(k-1) + 2\ln k - \ln(k+1)$	**M1 M1 A1**
	So, $\sum_{k=2}^{n} -\ln\left(1 - \frac{1}{k^2}\right) = \ln 2 + \ln n - \ln(n+1)$	**M1 A1**
	As $n \to \infty$, $\ln n - \ln(n+1) \to 0$	**B1**
	Therefore, $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{1^2} + \sum_{k=2}^{\infty} \frac{1}{k^2} < 1 + \ln 2$	**A1**

Question 1

Note that the statement of the question requires the use of a particular method in both parts.

(i)	
B1	Correct differentiation of the expression.
M1	Consideration of the sign of the derivative for positive values of x.
A1	Deduction that the derivative is positive for all positive values of x.
B1	Clear explanation that $x - \ln(1+x)$ is positive for all positive x. ***Note that answer is given in the question.***
B1	Use of $x = \frac{1}{k}$ and summation.
M1	Manipulation of logarithmic expression to form difference.
M1	Attempt to simplify the sum (some pairs cancelled out within sum).
A1	Clear explanation of result. ***Note that answer is given in the question.***
(ii)	
B1	Correct differentiation of the expression.
M1	Consideration of the sign of the derivative for $0 < x < 1$.
A1	Deduction that the derivative is negative for this range of values.
B1	Deduction that $x + \ln(1-x)$ is negative for this range of values.
B1	Use of $x = \frac{1}{k^2}$ and summation.
M1	Expression within logarithm as a single fraction and numerator simplified.
M1	Logarithm split to change at least one product to a sum of logarithms or one quotient as a difference of logarithms.
A1	Complete split of logarithm to required form.
M1	Use of differences to simplify sum.
A1	$\ln 2$ correct.
B1	Correctly dealing with limit as $n \to \infty$. **Note that answers which use ∞ as the upper limit on the sum from the beginning must have clear justification of the limit. Those beginning with n as the upper limit must have** $\ln n - \ln(n+1)$ correct in simplified sum.
A1	Inclusion of $k = 1$ to the sum to reach the final answer. ***Note that answer is given in the question.***

Question 2

	$\angle ACB = \pi - 3\alpha$	**B1**
	$\dfrac{AB}{\sin(\pi - 3\alpha)} = \dfrac{x}{\sin\alpha}$	**M1 A1**
	$\sin(\pi - 3\alpha) = \sin 3\alpha$	**B1**
	$\therefore AB = \dfrac{x\sin 3\alpha}{\sin\alpha}$	
	$= \dfrac{x(\sin\alpha\cos 2\alpha + \cos\alpha\sin 2\alpha)}{\sin\alpha}$	**M1**
	$= \dfrac{x(\sin\alpha \times (1 - 2\sin^2\alpha) + \cos\alpha \times 2\sin\alpha\cos\alpha)}{\sin\alpha}$	**M1 M1**
	$AB = (3 - 4\sin^2\alpha)x$	**A1**
	$DE = AB - BE - AD$ (or $DE = DB - BE$)	**B1**
	$DE = AB - BE - \frac{1}{2}AB = \frac{1}{2}AB - BE$	
	$DE = \dfrac{x}{2}(3 - 4\sin^2\alpha) - x\cos 2\alpha$	**B1 B1**
	$DE = \dfrac{x}{2}\left((3 - 4\sin^2\alpha) - 2(1 - 2\sin^2\alpha)\right) = \dfrac{x}{2}$	**M1 M1** **A1**
	$\sin(\angle FCE) = \frac{DE}{x} = \frac{1}{2}$, so $\angle FCE = \frac{\pi}{6}$	**B1** **M1 A1**
	Therefore $\angle ACF = \pi - 3\alpha - \left(\frac{\pi}{2} - 2\alpha\right) - \frac{\pi}{6} = \frac{\pi}{3} - \alpha$	**M1 M1**
	$\angle ACF = \dfrac{1}{3}(\pi - 3\alpha) = \dfrac{1}{3}\angle ACB$ So FC trisects the angle ACB	**A1**

Question 2

B1	Expression for $\angle ACB$ (may be implied by later working).
M1	Application of the sine rule.
A1	Correct statement.
B1	Does not need to be stated as long as implied in working.
M1	Use of $\sin(A+B)$ formula.
M1	Use of double angle formula for sin.
M1	Use of double angle formula for cos.
A1	Simplification of expression. ***Note that answer is given in the question.***
B1	Identification of this relationship between distances. (just $BD - BE$ is sufficient)
B1	Correct expression substituted for the length of BD.
B1	Correct expression substituted for the length of BE.
M1	Use of double angle formula for cos.
M1	Simplification of expression obtained.
A1	Correct expression independent of x.
B1	Identification of a right angled triangle to calculate $\sin(\angle FCE)$.
M1	Deduction that one of the lengths in sine of this angle is equal to DE.
A1	Value of the angle (Degrees or radians are both acceptable).
M1	Obtaining $\angle BCE = \frac{\pi}{2} - 2\alpha$
M1	Use of $\angle ACF = \angle ACB - \angle BCE - \angle FCE$.
A1	Expression to show that $\angle ACF = \frac{1}{3}\angle ACB$ and conclusion stated.

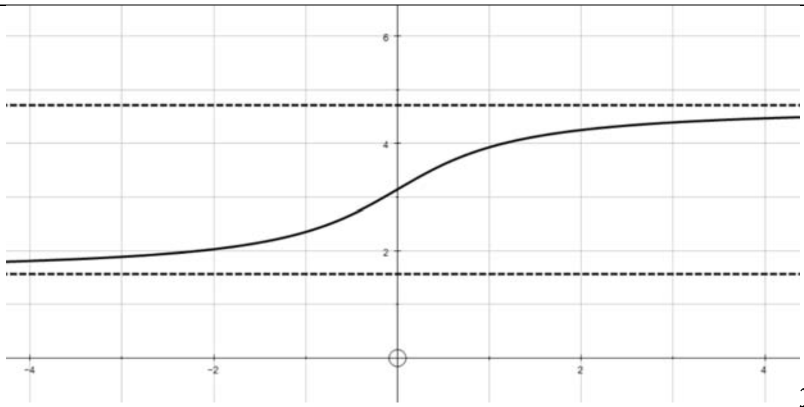
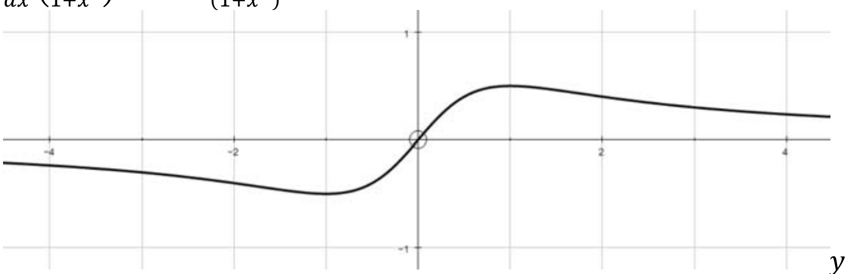
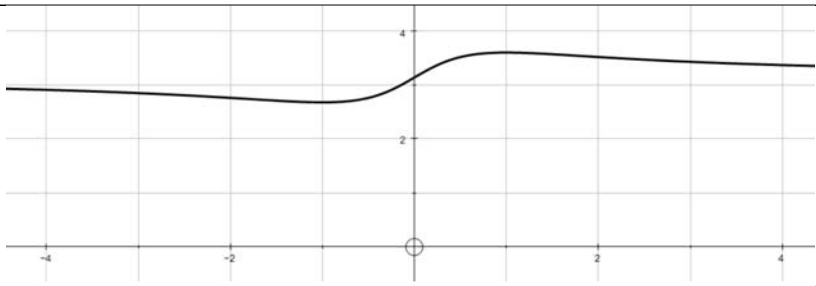
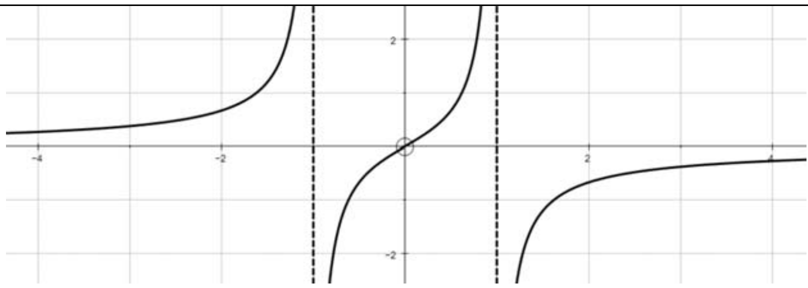
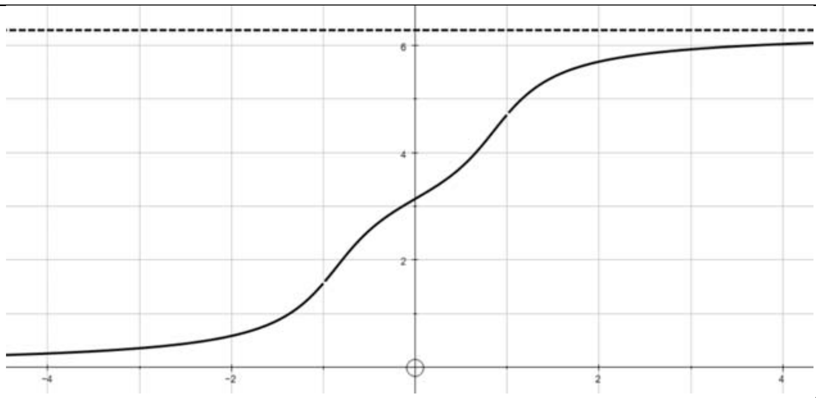
Question 3

	$T_8 - T_7$ is the number of triangles that can be made using a rod of length 8 and two other, shorter rods.	**M1**
	If the middle length rod has length 7 then the other rod can be 1, 2, 3, 4, 5 or 6.	**M1**
	If the middle length rod has length 6 then the other rod can be 2, 3, 4 or 5.	
	If the middle length rod has length 5 then the other rod can be 3 or 4.	**M1**
	$T_8 - T_7 = 2 + 4 + 6$.	**A1**
	Assume that the longest of the three rods has length 7:	**M1**
	If the middle length rod has length 6 then the other rod can be 1, 2, 3, 4 or 5.	**M1**
	If the middle length rod has length 5 then the other rod can be 2, 3 or 4.	
	If the middle length rod has length 4 then the other rod must be 3.	**M1**
	Therefore $T_7 - T_6 = 1 + 3 + 5$.	**A1**
	$T_8 - T_6 = T_8 - T_7 + T_7 - T_6 = 1 + 2 + 3 + 4 + 5 + 6$.	**A1**
	$T_{2m} - T_{2m-1} = 2 + 4 + \cdots + 2(m-1)$	**B1**
	$T_{2m} - T_{2m-2} = 1 + 2 + 3 + \cdots + 2(m-1)$	**B1**
	$T_4 = 3$. (The possibilities are $\{1,2,3\}, \{1,3,4\}$ and $\{2,3,4\}$.)	**B1**
	Substituting $m = 2$ into the equation gives $T_4 = \frac{1}{6}(2)(2-1)(4 \times 2 + 1) = 3$.	
	Therefore the formula is correct in the case $m = 2$.	**B1**
	Assume that the formula is correct in the case $m = k$: $T_{2(k+1)} = T_{2k} + \sum_{r=1}^{2k} r$	**M1**
	$T_{2(k+1)} = \frac{1}{6}k(k-1)(4k+1) + \frac{2k}{2}(2k+1)$	**M1**
	$T_{2(k+1)} = \frac{k}{6}[4k^2 - 3k - 1 + 12k + 6] = \frac{(k+1)}{6}(k)(4(k+1)+1)$, which is a statement of the formula where $m = k + 1$.	**M1**
	Therefore, by induction, $T_{2m} = \frac{1}{6}m(m-1)(4m+1)$	**A1**
	$T_{2m} - T_{2m-1} = 2 + 4 + \cdots + 2(m-1) = m(m-1)$.	**M1 A1**
	Therefore $T_{2m-1} = \frac{1}{6}m(m-1)(4m+1) - m(m-1)$.	
	$T_{2m-1} = \frac{1}{6}m(m-1)(4m-5)$. (Or $T_{2m+1} = \frac{1}{6}m(m+1)(4m-1)$)	**A1**

Question 3

M1	Appreciation of the meaning of $T_8 - T_7$.
M1	Identify the number of possibilities for the length of the third rod in one case.
M1	Identify the set of possible cases and find numbers of possibilities for each.
A1	Clear explanation of the result. ***Note that answer is given in the question.***
M1	An attempt to work out $T_7 - T_6$.
M1	Correct calculation for any one defined case.
M1	Identification of a complete set of cases.
A1	Correct value for $T_7 - T_6$.
A1	Correct deduction of expression for $T_8 - T_6$.
B1	Correct expression. No justification is needed for this mark.
B1	Correct expression. No justification is needed for this mark.
B1	Correct justification that $T_4 = 3$. Requires sight of possibilities or other justification.
B1	Evidence of checking a base case. (Accept confirmation that $m = 1$ gives $T_2 = 0$ here.)
M1	Application of the previously deduced result.
M1	Substitution of formula for $m = k$ and the formula for the sum.
M1	Taking common factor to give a single product.
A1	Re-arrangement to show that it is a statement of the required formula when $m = k + 1$ and conclusion stated.
M1	Use of result from start of question.
A1	Correct summation of $2 + 4 + \cdots + 2(m - 1)$.
A1	Correct formula reached (any equivalent expression is acceptable).

Question 4

(i)	 $y = f(x)$	**B1** **B1**
(ii)	$\frac{d}{dx}\left(\frac{x}{1+x^2}\right) = \frac{(1+x^2)(1)-(x)(2x)}{(1+x^2)^2}$, so stationary points when $x = \pm 1$.  $y = \frac{x}{1+x^2}$	**B1** **B1** **M1** **A1** **A1**
	 $y = g(x)$	**B1** **B1** **B1** **B1**
(iii)	 $y = \frac{x}{1-x^2}$	**B1** **B1** **B1** **B1**
	 $y = h(x)$	**B1** **B1** **B1** **B1** **B1**

Question 4

Penalise additional sections to graphs (vertical translations by $\pm\pi$) only on the first occasion providing that the correct section is present in later parts.

B1	Correct shape.
B1	Asymptotes $y = \frac{\pi}{2}$ and $y = \frac{3\pi}{2}$ shown.
B1	Rotational symmetry about the point $(0,0)$.
B1	Correct shape.
M1	Differentiation to find stationary points.
A1	Correct stationary points - $\left(\pm 1, \pm\frac{1}{2}\right)$. ($x$-coordinates)
A1	Correct y-coordinates.
B1	Rotational symmetry about the point $(0,\pi)$.
B1	Correct shape.
B1	Stationary points have same x-coordinates as previous graph. (Follow through incorrect stationary points in previous graph if consistent here).
B1	Correct co-ordinates for stationary points - $\left(\pm 1, \pi \pm \arctan\frac{1}{2}\right)$
B1	Correct asymptotes $x = \pm 1$.
B1	x-axis as an asymptote.
B1	Middle section correct shape.
B1	Outside sections correct shape.
B1	Section for $-1 < x < 1$ correct shape.
B1	$h(-1) = \frac{\pi}{2}$.
B1	$h(1) = \frac{3\pi}{2}$.
B1	Section for $x > 1$ correct with asymptote $y = 2\pi$.
B1	Section for $x > -1$ correct with asymptote $y = 0$ or a rotation of $x > 1$ section about $(0,\pi)$.

Question 5

(i)	$\tan S_1 = \tan\left(\arctan\frac{1}{2}\right) = \frac{1}{1+1}$, so the formula is correct for $n = 1$.	**B1**
	Assume that $\tan S_k = \frac{k}{k+1}$:	
	$S_{k+1} = S_k + \arctan\frac{1}{2(k+1)^2}$.	**M1**
	$\tan S_{k+1} = \frac{\frac{k}{k+1}+\frac{1}{2(k+1)^2}}{1-\left(\frac{k}{k+1}\right)\left(\frac{1}{2(k+1)^2}\right)}$	**M1**
	$\tan S_{k+1} = \frac{2k(k+1)^2+(k+1)}{2(k+1)^3-k}$, which simplifies to $\tan S_{k+1} = \frac{(k+1)}{(k+1)+1}$.	**M1 A1**
	Hence, by induction $\tan S_n = \frac{n}{n+1}$.	**A1**
	Clearly, $S_1 = \arctan\left(\frac{1}{2}\right)$.	**B1**
	Suppose that it is not true that $S_n = \arctan\left(\frac{n}{n+1}\right)$ for all values of n. Then there is a smallest positive value, k such that $S_k \neq \arctan\left(\frac{k}{k+1}\right)$.	
	Since $S_k > S_{k-1}$, $S_{k-1} = \arctan\left(\frac{k-1}{k}\right)$ and $\tan S_k = \frac{k}{k+1}$, but $S_k \neq \arctan\left(\frac{k}{k+1}\right)$ $S_k - S_{k-1} > \pi$.	**M1 M1**
	However, $S_k - S_{k-1} = \arctan\left(\frac{1}{2k^2}\right) < \frac{\pi}{2}$, so this is not possible.	**A1**
	Therefore $S_n = \arctan\left(\frac{n}{n+1}\right)$.	**A1**
(ii)	$\tan 2\alpha_n = \frac{4n^2}{4n^4-1}$.	**M1 A1**
	So, $\frac{2\tan\alpha_n}{1-\tan^2\alpha_n} = \frac{4n^2}{4n^4-1}$	**B1**
	Which simplifies to $2n^2\tan^2\alpha_n - (1-4n^2)\tan\alpha_n - 2n^2 = 0$	**M1 A1**
	$(\tan\alpha_n + 2n^2)(2n^2\tan\alpha_n - 1) = 0$	**A1**
	Since α_n must be acute, $\tan\alpha_n$ cannot equal $-2n^2$.	**B1**
	Therefore $\alpha_n = \arctan\left(\frac{1}{2n^2}\right)$.	
	$\sum_{n=1}^{\infty}\alpha_n = \lim_{n\to\infty} S_n = \arctan 1 = \frac{\pi}{4}$.	**M1 A1**

Question 5

B1	Confirmation that the formula is correct for $n = 1$.
M1	Expression of S_{k+1} in terms of S_k.
M1	Use of $\tan(A + B)$ formula.
M1	Simplification of fraction.
A1	Expression of S_{k+1} to show that it matches result.
A1	Conclusion stated.
B1	Confirmation for $n = 1$.
M1	Observation that $S_k - S_{k-1} > 0$
M1	Evidence of understanding that successive values of x with the same value of $\tan x$ must differ by π.
A1	Evidence of understanding that $S_k - S_{k-1}$ cannot be sufficiently large for S_k to be of the form $\arctan x$ if S_{k-1} is.
A1	Clear justification.
M1	Identification of the relevant sides of the triangle (diagram is sufficient).
A1	Correct expression for $\tan 2\alpha_n$.
B1	Use of double angle formula.
M1	Rearrangement to remove fractions.
A1	Correct quadratic reached.
A1	Quadratic factorised.
B1	Irrelevant case eliminated (must be justified).
M1	Sum expressed as limit of S_n
A1	Correct value justified. ***Note that answer is given in the question.***

Question 6

(i)	$\sec^2\left(\frac{\pi}{4}-\frac{x}{2}\right)=\frac{1}{\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}$	**B1**
	$\sec^2\left(\frac{\pi}{4}-\frac{x}{2}\right)=\frac{1}{\left(\cos\frac{\pi}{4}\cos\frac{x}{2}+\sin\frac{\pi}{4}\sin\frac{x}{2}\right)^2}$	**B1**
	$\sec^2\left(\frac{\pi}{4}-\frac{x}{2}\right)=\frac{2}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}$	
	$\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2\equiv\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}+\sin^2\frac{x}{2}=1+\sin x$	**M1**
	Therefore, $\sec^2\left(\frac{\pi}{4}-\frac{x}{2}\right)=\frac{2}{1+\sin x}$	**M1 A1**
	Therefore, $\int\frac{1}{1+\sin x}\,dx=-\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)+c$.	**M1 A1**
(ii)	Limits: $x=\pi\to y=0$ $x=0\to y=\pi$	
	$\frac{dy}{dx}=-1$	**B1**
	$\sin(\pi-x)=\sin x$	**B1**
	Therefore, $\int_0^\pi xf(\sin x)\,dx=\int_\pi^0(\pi-y)\,f(\sin(\pi-y))(-1)dy$	
	$\int_0^\pi xf(\sin x)\,dx=\int_0^\pi(\pi-x)\,f(\sin x)dx$	
	So, $2\int_0^\pi xf(\sin x)\,dx=\pi\int_0^\pi f(\sin x)dx$	**M1**
	$\int_0^\pi xf(\sin x)\,dx=\frac{\pi}{2}\int_0^\pi f(\sin x)dx$	**A1**
	$\int_0^\pi\frac{x}{1+\sin x}\,dx=\frac{\pi}{2}\int_0^\pi\frac{1}{1+\sin x}\,dx$, and applying the result from part (i):	
	$\int_0^\pi\frac{1}{1+\sin x}\,dx=\left[-\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right]_0^\pi=\left(-\tan\left(-\frac{\pi}{4}\right)\right)-\left(-\tan\left(\frac{\pi}{4}\right)\right)=2$.	**B1**
	$\int_0^\pi\frac{x}{1+\sin x}\,dx=\frac{\pi}{2}(2)=\pi$	**B1**
(iii)	Consider $\int_0^\pi x^3f(\sin x)\,dx$. Making the substitution $y=\pi-x$:	
	$\int_0^\pi x^3f(\sin x)\,dx=\int_\pi^0(\pi-y)^3\,f(\sin(\pi-y))(-1)dy$	**M1 A1**
	So, $\int_0^\pi x^3f(\sin x)\,dx=\int_0^\pi(\pi-x)^3\,f(\sin x)dx$	
	Therefore, $\int_0^\pi(2x^3-3\pi x^2)f(\sin x)\,dx=\int_0^\pi(\pi^3-3\pi^2x)\,f(\sin x)dx$	**B1**
	$\int_0^\pi\frac{1}{(1+\sin x)^2}\,dx=\frac{1}{4}\int_0^\pi\sec^4\left(\frac{\pi}{4}-\frac{x}{2}\right)dx$ $\int_0^\pi\frac{1}{(1+\sin x)^2}\,dx=\frac{1}{4}\int_0^\pi\sec^2\left(\frac{\pi}{4}-\frac{x}{2}\right)+\tan^2\left(\frac{\pi}{4}-\frac{x}{2}\right)\sec^2\left(\frac{\pi}{4}-\frac{x}{2}\right)dx$	**M1**
	$\int_0^\pi\frac{1}{(1+\sin x)^2}\,dx=\frac{1}{4}\left[-2\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)-\frac{2}{3}\tan^3\left(\frac{\pi}{4}-\frac{x}{2}\right)\right]_0^\pi=\frac{4}{3}$	**A1**
	And so, $\int_0^\pi\frac{x}{(1+\sin x)^2}\,dx=\frac{2\pi}{3}$	**B1**
	Therefore, $\int_0^\pi\frac{2x^3-3\pi x^2}{(1+\sin x)^2}\,dx=\pi^3\left(\frac{4}{3}\right)-3\pi^2\left(\frac{2\pi}{3}\right)=-\frac{2}{3}\pi^3$.	**B1**

Question 6

B1	Expression of $\sec^2\theta$ in terms of any other trigonometric functions.
B1	Correct use of a formula such as that for $\cos(A+B)$ to obtain expression with trigonometric functions of $\frac{x}{2}$.
M1	Expanding the squared brackets.
M1	Use of $\sin x \equiv 2\sin\frac{1}{2}x\cos\frac{1}{2}x$ and $\sin^2\frac{1}{2}x + \cos^2\frac{1}{2}x \equiv 1$
A1	Fully justified answer. ***Note that answer is given in the question.***
M1	Any multiple of $\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)$.
A1	Correct answer
B1	Deals with change of limits correctly. **AND** Correctly deals with change to integral with respect to u. **Note that both these steps need to be seen – the correct result reached without evidence of these steps should not score this mark.**
B1	Use of $\sin(\pi - x) = \sin x$ (may be just seen within working)
M1	Grouping similar integrals.
A1	Fully justified answer. ***Note that answer is given in the question.***
B1	Evaluation of the integral from (i) with the appropriate limits.
B1	Use of result from (ii) to evaluate required integral.
M1	Attempt to make the substitution.
A1	Substitution all completed correctly.
B1	Rearrange to give something that can represent the required integral on one side.
M1	Use of $\sec^2\theta \equiv 1 + \tan^2\theta$ within integral.
A1	Correct evaluation of this integral.
B1	Correct use of result from part (i).
B1	Correct application of result deduced earlier to reach final answer.

Question 7

(i)	Most likely examples: $x^2+\left(y\pm\sqrt{r^2-a^2}\right)^2=r^2$ and $\left(x\pm\sqrt{r^2-a^2}\right)^2+y^2=r^2$	**M1 M1 A1**
	If $r<a$ then there cannot be two points on the circle that are a distance of $2a$ apart and any two diametrically opposite points on C must be a distance of $2a$ apart.	**B1**
	If $r=a$ then the circle must be the same as C, so there is not exactly 2 points of intersection.	**B1**
(ii)	The distances of the centre of D from the centres of C_1 and C_2 are $\sqrt{r^2-{a_1}^2}$ and $\sqrt{r^2-{a_2}^2}$.	**M1 A1 B1**
	If the x-coordinate of the centre of D is x, then the y-coordinate is given by $r^2-{a_1}^2=y^2+(d+x)^2$ and $r^2-{a_2}^2=y^2+(d-x)^2$	**B1 B1**
	Therefore, $(d+x)^2-(d-x)^2=(r^2-{a_1}^2)-(r^2-{a_2}^2)$	**M1**
	$4dx={a_2}^2-{a_1}^2$ and so $x=\frac{{a_2}^2-{a_1}^2}{4d}$.	**M1 A1**
	Therefore, the y-coordinate of the centre of D satisfies $y^2=r^2-{a_1}^2-\left(d+\frac{{a_2}^2-{a_1}^2}{4d}\right)^2$ and $y^2=r^2-{a_2}^2-\left(d-\frac{{a_2}^2-{a_1}^2}{4d}\right)^2$	**B1**
	So $2y^2=2r^2-{a_1}^2-{a_2}^2-\left(d+\frac{{a_2}^2-{a_1}^2}{4d}\right)^2-\left(d-\frac{{a_2}^2-{a_1}^2}{4d}\right)^2$	
	$2y^2=2r^2-{a_1}^2-{a_2}^2-2d^2-2\left(\frac{{a_2}^2-{a_1}^2}{4d}\right)^2$	
	So, $y=\sqrt{r^2-\frac{{a_1}^2+{a_2}^2}{2}-d^2-\left(\frac{{a_2}^2-{a_1}^2}{4d}\right)^2}$	
	Therefore, $r^2-\frac{{a_1}^2+{a_2}^2}{2}-d^2-\left(\frac{{a_2}^2-{a_1}^2}{4d}\right)^2\geq 0$	**B1**
	$16r^2d^2-8{a_1}^2d^2-8{a_2}^2d^2-16d^4-({a_2}^2-{a_1}^2)^2\geq 0$	**M1 M1**
	$16r^2d^2\geq 16d^4+8{a_1}^2d^2+8{a_2}^2d^2+({a_2}^2-{a_1}^2)^2$	
	$16r^2d^2\geq(4d^2+{a_1}^2+{a_2}^2)^2+({a_2}^2-{a_1}^2)^2-({a_1}^2+{a_2}^2)^2$	**M1**
	$16r^2d^2\geq(4d^2+{a_1}^2+{a_2}^2)^2-4{a_1}^2{a_2}^2$	**M1**
	$16r^2d^2\geq(4d^2+{a_1}^2+{a_2}^2-2a_1a_2)(4d^2+{a_1}^2+{a_2}^2+2a_1a_2)$	
	$16r^2d^2\geq(4d^2+(a_1-a_2)^2)(4d^2+(a_1+a_2)^2)$	**A1**

Question 7

M1	Calculation that the distance between the centres of the circles must be $\sqrt{r^2 - a^2}$.
M1	An example which shows that it is possible for at least one value of r.
A1	Example showing that it is possible for all $r > a$.
B1	Statement that the two intersection points must be a distance $2a$ apart.
B1	Explanation that in the case $r = a$ it would have to be the same circle.
M1	The line joining the centre of C_1 (or C_2) and the radii to a point of intersection form a right angled triangle in each case. (one case)
A1	Use of this to find the distance between centres of circles.
B1	Applying the same to the other circle.
B1	Expression relating the co-ordinates and radii obtained from considering C_1.
B1	Expression relating the co-ordinates and radii obtained from considering C_2.
M1	Elimination of y from the equations.
M1	Either expansion of squared terms or rearrangement to apply difference of two squares.
A1	Expression for x reached. ***Note that answer is given in the question.***
B1	Substitution to find expression for y-coordinate. ***Note that any expression for y in terms of d, r, a_1 and a_2 is sufficient, but it must be expressed as $y = \cdots$, not $y^2 = \cdots$.***
B1	Observation that y^2 must be positive.
	Alternative mark scheme for this may be required once some solutions seen.
M1	Attempt to rearrange the inequality to get $16r^2d^2$ on the left.
M1	Reach a point symmetric in a_1 and a_2.
M1	Reach a combination of squared terms.
M1	Apply difference of two squares to simplify.
A1	Reach the required inequality. ***Note that answer is given in the question.***

Question 8

(i)	Let $\boldsymbol{a}$ be the vector from the centre of C_2 to P.	
	Using similar triangles, the vector from the centre of C_1 to P is $\frac{r_1}{r_2}\boldsymbol{a}$.	**M1 A1**
	Therefore $\frac{r_1}{r_2}\boldsymbol{a} - \boldsymbol{a} = \boldsymbol{x_2} - \boldsymbol{x_1}$, since these are both expressions for the vector from the centre of C_1 to the centre of C_2.	**M1**
	So $\boldsymbol{a} = \frac{r_2}{r_1 - r_2}(\boldsymbol{x_2} - \boldsymbol{x_1})$	**A1**
	The position vector of P is $\boldsymbol{x_2} + \frac{r_2}{r_1 - r_2}(\boldsymbol{x_2} - \boldsymbol{x_1}) = \frac{r_1\boldsymbol{x_2} - r_2\boldsymbol{x_1}}{r_1 - r_2}$	**M1 A1**
(ii)	The position vectors of Q and R will be $\frac{r_3\boldsymbol{x_1} - r_1\boldsymbol{x_3}}{r_3 - r_1}$ and $\frac{r_2\boldsymbol{x_3} - r_3\boldsymbol{x_2}}{r_2 - r_3}$.	**B1**
	Therefore, $\overrightarrow{PQ} = \frac{r_3\boldsymbol{x_1} - r_1\boldsymbol{x_3}}{r_3 - r_1} - \frac{r_1\boldsymbol{x_2} - r_2\boldsymbol{x_1}}{r_1 - r_2} = \frac{\boldsymbol{x_1}[r_3(r_1 - r_2) + r_2(r_3 - r_1)] - \boldsymbol{x_2}[r_1(r_3 - r_1)] - \boldsymbol{x_3}[r_1(r_1 - r_2)]}{(r_3 - r_1)(r_1 - r_2)}$	**M1 A1**
	$\overrightarrow{PQ} = \frac{r_1}{(r_3 - r_1)(r_1 - r_2)}(\boldsymbol{x_1}[r_3 - r_2] + \boldsymbol{x_2}[r_1 - r_3] + \boldsymbol{x_3}[r_2 - r_1])$	**M1 A1**
	Similarly, $\overrightarrow{QR} = \frac{r_3}{(r_2 - r_3)(r_3 - r_1)}(\boldsymbol{x_1}[r_3 - r_2] + \boldsymbol{x_2}[r_1 - r_3] + \boldsymbol{x_3}[r_2 - r_1])$	**M1 A1** **M1 A1**
	Since they are multiples of each other the points P, Q and R must lie on the same straight line.	**B1**
(iii)	Q lies halfway between P and R if $\overrightarrow{PQ} = \overrightarrow{QR}$	**B1**
	Therefore $\frac{r_1}{(r_3 - r_1)(r_1 - r_2)} = \frac{r_3}{(r_2 - r_3)(r_3 - r_1)}$	**M1**
	So, $r_1(r_2 - r_3) = r_3(r_1 - r_2)$	
	Which simplifies to $r_1 r_2 + r_2 r_3 = 2r_1 r_3$	**M1 A1**

Question 8

M1	Identification of similar triangles within the diagram.
A1	Relationship between the two vectors to P.
M1	Equating two expressions for the vector between the centres of the circles.
A1	Correct simplified expression.
M1	Calculation of vector from centre of one circle to P.
A1	Correct position vector for P. ***Note that answer is given in the question.***
B1	Identifying the correct vectors for the foci of the other pairs of circles.
M1	Expression for vector between any two of the foci.
A1	Terms grouped by vector.
M1	Simplification of grouped terms.
A1	Extraction of common factor.
M1 A1 M1 A1	Expression for a vector between a different pair of foci. ***Award marks as same scheme for previous example, but award all four marks for the correct answer written down as it can be obtained by rotating 1, 2 and 3 in the previous answer.***
B1	Statement that they lie on a straight line.
B1	Statement that the two vectors must be equal.
M1	Reduction to statement involving only r terms.
M1	Attempt to simplify expression obtained (if necessary).
A1	Any simplified form.

Question 9

(i)	Taking moments about A:	
	$M_B = 3mga\sin(30+\theta)$	**M1 A1**
	$M_C = 5mga\sin(30-\theta)$	**M1 A1**
	$M_B = M_C$	**B1**
	$5mga(\cos 30\sin\theta + \cos\theta\sin 30) = 3mga(\cos 30\sin\theta - \cos\theta\sin 30)$	**M1**
	$5\left(\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta\right) = 3\left(\frac{\sqrt{3}}{2}\sin\theta - \frac{1}{2}\cos\theta\right)$	**A1**
	Therefore $4\sqrt{3}\sin\theta = \cos\theta$	**A1**
	Either Use $\sin^2\theta + \cos^2\theta \equiv 1$ and justify choice of positive square root. **Or** Draw right angled triangle such that $\tan\theta = \frac{1}{4\sqrt{3}}$ and calculate the length of the hypotenuse.	**M1**
	$\sin\theta = \frac{1}{7}$	**A1**
(ii)	Let h_1 be the vertical distance of B below A. Let h_2 be the vertical distance of C below A.	
	$h_1 = a\sin\left(\frac{\pi}{3} - \theta\right) = \frac{11}{14}a$	**M1 M1 A1**
	$h_2 = a\sin\left(\frac{\pi}{3} + \theta\right) = \frac{13}{14}a$	**M1 A1**
	If X is the centre of mass of the triangle: $AX = h = \frac{3h_1+5h_2}{8} = \frac{7}{8}a$	**M1 A1**
	Conservation of energy: $4mv^2 \geq 8mg.2h$ for complete revolutions.	**M1 A1**
	Therefore $v_0 = \sqrt{\frac{7ga}{2}}$.	**A1**

Question 9

M1	Attempt to find the moment of B about A.
A1	Correct expression for moment ($\sin(30+\theta)$ may be replaced by $\cos(60-\theta)$).
M1	Attempt to find the moment of C about A.
A1	Correct expression for moment ($\sin(30-\theta)$ may be replaced by $\cos(60+\theta)$).
B1	Correct statement that these must be equal.
M1	Use of $\sin(A \pm B)$ or $\cos(A \pm B)$ formulae.
A1	Correct values used for $\sin 30$ and $\cos 30$.
A1	Correctly simplified.
M1	Use of a correct method to find the value of $\sin\theta$.
A1	Fully justified solution. If using right angled triangle method then choice of positive root not needed, if choice of positive root not given when applying $\sin^2\theta + \cos^2\theta \equiv 1$ method then M1 A0 should be awarded. ***Note that answer is given in the question.***
M1	Attempt to find h_1.
M1	Correctly deal with sine or cosine term.
A1	Correct value.
M1	Attempt to find h_2.
A1	Correct value.
M1	Combine two values to obtain distance of centre of mass from A.
A1	Correct value
M1	Apply conservation of energy.
A1	Correct inequality.
A1	Correct minimum value.

Question 9 Alternative part (i)

(i)	Let X be the centre of mass of the triangle and let the distance CX be d.	
	Taking moments about X: $5mgd\cos\theta = 3mg(a-d)\cos\theta$	**M1 A1**
	Therefore $5d = 3(a-d)$, so $d = \frac{3}{8}a$.	**A1**
	X must lie on BC and $\angle XAC = 30 - \theta$.	**B1**
	$\sin(30-\theta) = \dfrac{\frac{3}{8}a\cos\theta}{a}$	**M1**
	$\sin 30\cos\theta + \cos 30\sin\theta = \frac{3}{8}\cos\theta$	**M1**
	$\dfrac{\cos\theta}{8} = \dfrac{\sqrt{3}\sin\theta}{2}$.	**A1**
	Therefore $\cos\theta = 4\sqrt{3}\sin\theta$ and so $\cos^2\theta = 48\sin^2\theta$	**M1**
	$\sin^2\theta = \frac{1}{49}$, and so (since θ is acute) $\sin\theta = \frac{1}{7}$.	**M1 A1**

M1	Taking moments.
A1	Correct equation.
A1	Correct relationship between d and a.
B1	Identification that X lies on BC and calculation of $\angle XAC$.
M1	Use of sine of identified angle.
M1	Use of $\sin(A-B)$ formula.
A1	Direct relationship between $\sin\theta$ and $\cos\theta$.
M1	Rearrangement and squaring both sides.
M1	Applying $\sin^2\theta + \cos^2\theta \equiv 1$.
A1	Final answer (choice of positive root must be explained). ***Note that answer is given in the question.***

Question 10

	If the length of string from the hole at any moment is l, then $\frac{dl}{dt} = -V$.	**B1**
	The distance, x, from the point beneath the hole satisfies, $h^2 + x^2 = l^2$.	**B1**
	Therefore $\frac{dx}{dt} = \frac{d}{dt}\left((l^2 - h^2)^{\frac{1}{2}}\right) = \frac{1}{2}(l^2 - h^2)^{-\frac{1}{2}} \times 2l\frac{dl}{dt}$.	**M1 A1**
	$\frac{dx}{dt} = -lV(l^2 - h^2)^{-\frac{1}{2}} = -V \times \frac{l}{x}$, and $\frac{l}{x} = \operatorname{cosec}\theta$	**M1**
	Therefore, the speed of the particle is $V \operatorname{cosec}\theta$.	**A1**
	Acceleration: $\frac{d}{dt}(V \operatorname{cosec}\theta) = -V \operatorname{cosec}\theta \cot\theta \times \frac{d\theta}{dt}$	**M1 A1**
	$\sin\theta = \frac{x}{l}$, so $\cos\theta\frac{d\theta}{dt} = \frac{l(-V\operatorname{cosec}\theta) - l\sin\theta(-V)}{l^2} = \frac{V(\sin^2\theta - 1)}{l\sin\theta}$	**M1**
	Therefore $\frac{d\theta}{dt} = -\frac{V}{l}\cot\theta$	**A1**
	The acceleration is $\frac{V^2}{l\sin\theta}\cot^2\theta$	**M1**
	Since $l = h\sec\theta$, the acceleration can be written as $\frac{V^2}{h}\cot^3\theta$.	**M1 A1**
	Horizontally: $T\sin\theta = \frac{mV^2}{h}\cot^3\theta$, so $T = \frac{mV^2}{h}\cot^3\theta\operatorname{cosec}\theta$	**M1 M1 A1**
	The particle will leave the floor when $T\cos\theta = mg$	**M1 A1**
	$\frac{mV^2}{h}\cot^4\theta = mg$ and so $\tan^4\theta = \frac{V^2}{gh}$	**M1 A1**

Question 10

B1	An interpretation of V in terms of other variables (including any newly defined ones).
B1	Any valid relationship between the variables.
M1	Differentiation to find horizontal velocity.
A1	Correct differentiation.
M1	Attempt to eliminate any introduced variables.
A1	Correct result. ***Answers which make clear reference to the speed of the particle in the direction of the string being V.***
M1	Differentiation of speed found in first part.
A1	Correct answer.
M1	Attempt to differentiate to find an expression for $\frac{d\theta}{dt}$.
A1	Correct answer.
M1	Substitution to find expression for acceleration.
M1	Relationship between required variables and any extra variables identified.
A1	Substitution to give answer in terms of correct variables.
M1	Horizontal component of tension.
M1	Application of Newton's second law.
A1	Correct answer.
M1	Vertical component of tension found.
A1	Identification that particle leaves ground when tension is equal to the mass.
M1	Substitution of their value for T.
A1	Rearrangement to give required result. ***Note that answer is given in the question.***

Question 11

(i)	$A(x - a\cos\theta\,, a\sin\theta)$	**B1 B1**
	Differentiating: $\left(\dot{x} - a(-\sin\theta)\dot{\theta}, a(\cos\theta)\dot{\theta}\right)$	**M1**
	Since B is moving with velocity v and is at the point $(x, 0)$ at time t, $\dot{x} = v$:	
	Velocity of A is $\left(v + a\dot{\theta}\sin\theta\,, a\dot{\theta}\cos\theta\right)$.	**A1**
(ii)	Initial momentum was mu (horizontally).	**M1**
	Horizontal velocity of C will be the same as that of A, so horizontally the total momentum is given by $mv + 2m\left(v + a\dot{\theta}\sin\theta\right)$	**M1**
	Therefore $3v + 2a\dot{\theta}\sin\theta = u$.	**A1**
	Initial energy was $\frac{1}{2}mu^2$	**M1**
	Total energy is $\frac{1}{2}mv^2 + 2\left(\frac{1}{2}m\left(\left(v + a\dot{\theta}\sin\theta\right)^2 + \left(a\dot{\theta}\cos\theta\right)^2\right)\right)$	**M1 A1**
	Therefore $u^2 = v^2 + 2\left(v^2 + 2av\dot{\theta}\sin\theta + a^2\dot{\theta}^2\sin^2\theta + a^2\dot{\theta}^2\cos^2\theta\right)$	**M1**
	So $u^2 = 3v^2 + 4av\dot{\theta}\sin\theta + 2a^2\dot{\theta}^2$	
	Substituting $v = \frac{u - 2a\dot{\theta}\sin\theta}{3}$ gives $3u^2 = \left(u - 2a\dot{\theta}\sin\theta\right)^2 + 4a\dot{\theta}\sin\theta\left(u - 2a\dot{\theta}\sin\theta\right) + 6a^2\dot{\theta}^2$	**M1**
	$6a^2\dot{\theta}^2 = 3u^2 - u^2 + 4au\dot{\theta}\sin\theta - 4a^2\dot{\theta}^2\sin^2\theta - 4au\dot{\theta}\sin\theta + 8a^2\dot{\theta}^2\sin^2\theta$	
	$6a^2\dot{\theta}^2 - 4a^2\dot{\theta}^2\sin^2\theta = 2u^2$	
	So, $\dot{\theta}^2 = \frac{u^2}{a^2(3 - 2\sin^2\theta)}$.	**A1**
(iii)	$\dot{\theta}^2 > 0$, so there can only be an instantaneous change of direction in which θ varies at a collision. Since the first collision will be when $\theta = 0$, the second collision must be when $\theta = \pi$.	**B1** **B1**
(iv)	Since horizontal momentum must be mu, $v = 0 \implies 2a\dot{\theta}\sin\theta = u$.	**B1**
	The KE of A must be $\frac{1}{4}mu^2$, so $\frac{1}{2}ma^2\dot{\theta}^2 = \frac{1}{4}mu^2$	**B1**
	$\frac{1}{2}ma^2\dot{\theta}^2 = ma^2\dot{\theta}^2\sin^2\theta$	
	$\sin^2\theta = \frac{1}{2}$, so $\theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$	**M1 A1**
	v is only 0 when θ takes these values and $\dot{\theta}$ is positive as v would need a non-zero value to satisfy $3v + 2a\dot{\theta}\sin\theta = u$ if $\dot{\theta}$ is negative. (The relationship is still true since collisions are elastic).	**B1**

Question 11

B1	Horizontal component.
B1	Vertical component.
M1	Differentiation.
A1	Complete justification, including clear explanation that $\dot{x} = v$. ***Note that answer is given in the question.***
M1	Statement that momentum will be conserved.
M1	Identification that horizontal momentum of A and C will be equal.
A1	Correct equation reached. ***Note that answer is given in the question.***
M1	Statement that energy will be conserved.
M1	Use of symmetry to obtain energy of C (accept answers which simply double the energy of A rather than stating the vertical velocity in opposite direction).
A1	Correct relationship.
M1	Use of $\sin^2\theta + \cos^2\theta \equiv 1$.
M1	Substituting the other relationship to eliminate v.
A1	Correct equation reached. ***Note that answer is given in the question.***
B1	Correct value of θ.
B1	Answer justified.
B1	First equation identified.
B1	Second equation identified.
M1	Solving simultaneously to find θ.
A1	Correct values for θ.
B1	Justified answer that v is not always 0 when θ takes these values.

Question 12

(i)	If a tail occurs then player B must always win before A can achieve the sequence required. Therefore the only way for A to win is if both of the first two tosses are heads.	**B1**
	After the first two tosses are heads it does not matter if more tosses result in heads as the first time tails occurs A will win.	**B1**
	The probability that A wins is therefore $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$	**B1**
(ii)	As before, after HH, only A can win.	**B1**
	Similarly, after TT, only C can win.	**B1**
	In all other cases for the first two tosses only B and D will be able to win.	**M1**
	The probabilities for B and D to win must be equal.	**M1**
	The probability of winning is $\frac{1}{4}$ for all of the players.	**A1**
(iii)	If the first two tosses are TT then C must win (as soon as a H occurs), so the probability is 1.	**B1**
	After HT: C must win if the next toss is a T as B needs two Hs to win, but C will win the next time an H occurs.	**M1**
	If the next toss is H, then the position is as if the first two tosses had been TH, and so the probability that C wins from this point is q.	**M1**
	Therefore, $p = \frac{1}{2} \times 1 + \frac{1}{2} \times q$	**A1**
	After HH: If the next toss is H then C will win with probability r.	
	If the next toss is T then C will win with probability p.	**M1**
	Therefore $r = \frac{1}{2}r + \frac{1}{2}p$, and so $p = r$.	**A1**
	After TH: If the next toss is H then player B wins immediately.	
	If the next toss is T then C will win with probability p.	**M1**
	Therefore $q = \frac{1}{2}p$.	**A1**
	Solving the two equations in p and q, gives $p = \frac{2}{3}$, $q = \frac{1}{3}$	
	From the third equation $r = \frac{2}{3}$	**M1 A1**
	The probability that C wins is $\frac{1}{4}\left(1 + \frac{2}{3} + \frac{1}{3} + \frac{2}{3}\right) = \frac{2}{3}$	**M1 A1**

Question 12

B1	Identifying that A cannot win once a tail has been tossed.
B1	Identifying that A must win once the first two tosses have been heads.
B1	Showing the calculation to reach the answer. ***Note that answer is given in the question.***
B1	Recognising that the situation is unchanged for player A.
B1	Recognising that the same logic applies to player C.
M1	All other cases lead to wins for one of the remaining players.
M1	Recognising that the probabilities must be equal.
A1	Correct statement of the probabilities. **If no marks possible by this scheme award one mark for each probability correctly calculated with supporting working. All four calculated scores 5 marks.**
B1	Explanation that probability must be 1.
M1	Explanation of the case that the next toss is T. **This mark and the next could be awarded for an appropriate tree diagram.**
M1	Explanation of the case that the next toss is H.
A1	Justification of the relationship between p and q. ***Note that answer is given in the question.***
M1	Consideration of one case following HH. **This mark could be awarded for an appropriate tree diagram.**
A1	Establishment of the relationship.
M1	Consideration of one case following TH. **This mark could be awarded for an appropriate tree diagram.**
A1	Establishment of the relationship.
M1	Attempt to solve the simultaneous equations.
A1	Correct values for p, q and r.
M1	Attempt to combine probabilities to obtain overall probability of win.
A1	Correct probability.

Question 13

(i)	$C = \begin{matrix} ky + a(X-y) & \text{for } X > y \\ ky & \text{for } X \le y \end{matrix}$	**B1**
	$E(C) = ky + a\int_y^\infty (x-y)\lambda e^{-\lambda x}dx$	**M1 M1 A1**
	Use the substitution $u = x - y$ in the integral:	
	$\int_y^\infty (x-y)\lambda e^{-\lambda x}dx = e^{-\lambda y}\int_0^\infty u\lambda e^{-\lambda u}du$	**B1**
	$\int_0^\infty u\lambda e^{-\lambda u}du = \left[-ue^{-\lambda u}\right]_0^\infty + \int_0^\infty e^{-\lambda u}du = \left[-ue^{-\lambda u} - \frac{1}{\lambda}e^{-\lambda u}\right]_0^\infty = \frac{1}{\lambda}$	**M1**
	Therefore $E(C) = ky + \frac{a}{\lambda}e^{-\lambda y}$.	**A1**
	$\frac{d}{dy}(E(C)) = k - ae^{-\lambda y}$, so the stationary point occurs when $y = \frac{1}{\lambda}\ln\frac{a}{k}$.	**M1 A1**
	If $\frac{a}{k} > 1$ then choose $y = \frac{1}{\lambda}\ln\frac{a}{k}$ as it is positive.	
	If $\frac{a}{k} \le 1$ then choose $y = 0$ as the minimum occurs at a negative value of y.	**B1**
(ii)	$E(C^2) = k^2y^2 + \int_y^\infty 2aky(x-y)\lambda e^{-\lambda x} + a^2(x-y)^2\lambda e^{-\lambda x}dx$	**M1 A1**
	Use the substitution $u = x - y$ in the integral:	
	$Integral = e^{-\lambda y}\int_0^\infty 2akyu\lambda e^{-\lambda u} + a^2u^2\lambda e^{-\lambda u}dx$	**B1**
	Applying integration done before: $\int_0^\infty 2akyu\lambda e^{-\lambda u}dx = \frac{2aky}{\lambda}$	
	Using integration by parts: $\int_0^\infty a^2u^2\lambda e^{-\lambda u}dx = \left[-a^2u^2e^{-\lambda u}\right]_0^\infty + \int_0^\infty \frac{2a^2u\lambda e^{-\lambda u}}{\lambda}dx$	**M1 A1**
	and, applying the integration already completed, $\int_0^\infty \frac{2a^2ue^{-\lambda u}}{\lambda}dx = \frac{2a^2}{\lambda^2}$.	
	Therefore $E(C^2) = k^2y^2 + \frac{2aky}{\lambda}e^{-\lambda y} + \frac{2a^2}{\lambda^2}e^{-\lambda y}$.	**A1**
	$\text{Var}(C^2) = E(C^2) - E(C)^2$	**M1**
	$\text{Var}(C^2) = k^2y^2 + \frac{2aky}{\lambda}e^{-\lambda y} + \frac{2a^2}{\lambda^2}e^{-\lambda y} - \left(ky + \frac{a}{\lambda}e^{-\lambda y}\right)^2$.	
	$\text{Var}(C^2) = \frac{a^2}{\lambda^2}\left(2e^{-\lambda y} - e^{-2\lambda y}\right)$.	**A1**
	$\frac{d}{dy}\left(\text{Var}(C^2)\right) = \frac{2a^2}{\lambda}e^{-\lambda y}\left(e^{-\lambda y} - 1\right)$	**M1**
	For $y > 0$, $\frac{d}{dy}\left(\text{Var}(C^2)\right) < 0$, so the variance decreases as y increases.	**A1**

Question 13

B1	Statement of random variable.
M1	Any correct term in expectation (allow ky multiplied by an attempt at the probability for not needing any extra costs).
M1	Correct integral stated (allow $-y$ missing).
A1	Fully correct statement.
	May be altered to accommodate other methods once solutions seen.
B1	Substitution performed correctly.
M1	Integration by parts used to calculate integral.
A1	Correctly justified solution. ***Note that answer is given in the question.***
M1	Differentiation to find minimum point.
A1	Correct identification of point.
B1	Both cases identified with the solutions stated.
M1	Attempt at $E(C^2)$ (at least two terms correct).
A1	Correct statement of $E(C^2)$.
B1	Substitution performed correctly.
M1	Applying integration by parts.
A1	Correct integration.
A1	Correct expression for $E(C^2)$.
M1	Use of $\mathrm{Var}(C^2) = E(C^2) - E(C)^2$
A1	Correct simplified form for $\mathrm{Var}(C^2)$
M1	Differentiation of $\mathrm{Var}(C^2)$.
A1	Correct interpretation. ***Note that answer is given in the question.***